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On buoyancy induced heat and mass transfer from a concentrated source in an infinite porous medium

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Abstract—This paper reports an analytical study of natural convection heat and mass transfer, induced by a concentrated source, located in an infinite porous medium. The transient and steady-state flow, temperature and concentration fields are obtained in terms of series expansions in the Rayleigh number based on the permeability of the porous medium and the heat generation rate from the source. The impact of the chemical species created by the source is to either aid or retard the flow induced by thermal buoyancy. Expressions determining the effect of species generation on the resulting transient and steady-state temperature and flow fields in the porous medium are reported in the course of the study. All discussions in this paper focus on the case where the net flow is upwards. Even though heat was specified to be one of the two diffusion mechanisms, the results of the present study apply as well to the case of buoyancy induced flow from a concentrated source generating simultaneously two different chemical components.

1. INTRODUCTION

BUOYANCY driven convection in fluid-saturated porous media is recently becoming a focal point of engineering heat transfer research. Among the thermal engineering applications which benefit from a better understanding of the fundamentals of heat and fluid flow in a porous medium are thermal insulations, geothermal systems, cooling of nuclear reactors and underground disposal of nuclear wastes.

With the exception of the problem of double diffusion in a horizontal porous layer [1–3], the majority of published studies on porous medium natural convection are concerned with the case where the driving buoyancy mechanism is induced by temperature gradients alone. However, very often chemical species concentration gradients greatly affect the buoyancy driven flow and, as a result, they play a decisive role on the development of the temperature field. Even though the importance of this class of problems has been established in the literature for classical fluids (see for example refs. [4–8]), the equivalent class of problems in porous media has been, to a great extent [9], overlooked.

The object of this paper is to analyze an important fundamental problem of penetrative convection in porous medium, namely the transient and steady-state flow, temperature and concentration patterns developing around a point source generating heat and, at the same time, a chemical species, when the natural convection flow is induced by the combined action of temperature and concentration gradients.

The practical significance of this problem can be exemplified by the spreading of a pollutant created by an exothermic reaction at an underground site. In addition to its importance in geophysics, the problem finds applications in the chemical industry.

Natural convection from a point heat source in porous medium in the absence of concentration gradients has been investigated theoretically by Bejan [10] and Wooding [11] for low and high Rayleigh numbers, respectively. The present study uses a theoretical approach similar to the one in [10], hence, the results reported here are representative of the low Rayleigh number regime.

In conclusion, the main goal of this article is to investigate analytically the effect of species diffusion on the buoyancy induced heat and fluid flow from a point source in an unbounded porous medium.

2. MATHEMATICAL FORMULATION

Consider a point source located at the origin of the spherical-polar coordinate system shown in Fig. 1. The source, which is surrounded by an unbounded fluid-saturated porous medium, generates heat at a rate q (W) and, at the same time, a substance at a rate m (kg s^{-1}). The density of the generated substance is different from the density of the fluid saturating the porous medium. Since the vertical axis is parallel to the gravity vector (Fig. 1) the problem is symmetric in the angular direction ϕ.

According to the Darcy flow model [12, 13] the equations describing the conservation of mass, momentum, energy and species at each point in the porous medium are

$$\frac{\partial}{\partial r}(r^2 u \sin\theta) + \frac{\partial}{\partial\theta}(rv\sin\theta) = 0 \qquad (1)$$

NOMENCLATURE

A	dimensionless parameter, equation (25)
B_i	constants of integration, equations (36), (37)
c	species concentration [kg m^{-3}]
c_p	specific heat of fluid at constant pressure
D	mass diffusivity [m^2 s^{-1}]
D_i	coefficients in equations (39), (41)
E_i	coefficients in equations (40), (42)
f	function used for separation of variables, equation (28)
F	function used for separation of variables, equation (33)
G	function used for separation of variables, equation (32)
g	gravitational acceleration
k	thermal conductivity of fluid/porous matrix composite
K	permeability of porous medium
Le	Lewis number, equation (16)
m	rate of species generation from the source [kg s^{-1}]
N	dimensionless parameter, equation (17)
P	pressure
q	rate of heat generation from the source [W]
r	radial coordinate, Fig. 1
R	given radial distance
Ra	Darcy-modified Rayleigh number based on the permeability of the porous matrix and the heat generation rate at the source, equation (14)
T	temperature
t	time
u	radial velocity, Fig. 1
v	tangential velocity, Fig. 1.

Greek symbols

α	thermal diffusivity, $k/(\rho c_p)_f$
β	coefficient of thermal expansion
β_c	coefficient of concentration expansion
γ	dimensionless parameter, equation (15)
η	similarity variable, equation (24)
θ	angle, Fig. 1
λ	porosity of porous matrix
μ	viscosity
ν	kinematic viscosity, μ/ρ
ρ	fluid density
σ	heat capacity ratio
ϕ	angle, Fig. 1
ψ	streamfunction.

Subscripts

$*$	pertaining to dimensionless quantities
∞	denoting a reference state.

$$u = -\frac{K}{\mu}\left(\frac{\partial P}{\partial r} + \rho g \cos\theta\right) \tag{2}$$

$$v = -\frac{K}{\mu}\left[\frac{1}{r}\frac{\partial P}{\partial \theta} - \rho g \sin\theta\right] \tag{3}$$

$$\sigma\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)\right] \tag{4}$$

$$\lambda\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r} + \frac{v}{r}\frac{\partial c}{\partial \theta} = D\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial c}{\partial \theta}\right)\right]. \tag{5}$$

In the above equations u and v are the radial and the tangential velocity components, T is the temperature, c the concentration of the substance generated by the source, P the pressure, t the time, ρ the fluid density, μ the fluid viscosity, D the species diffusivity in the porous medium when filled with fluid and α the thermal diffusivity of the porous medium, which is equal to the ratio of the thermal conductivity of the porous medium filled with stagnant fluid, k, divided by the heat capacity of the fluid, $(\rho c_p)_f$. Following the Darcy model of flow the porous medium is assumed homogeneous of permeability K and porosity λ. Parameter σ stands for the heat capacity ratio defined as

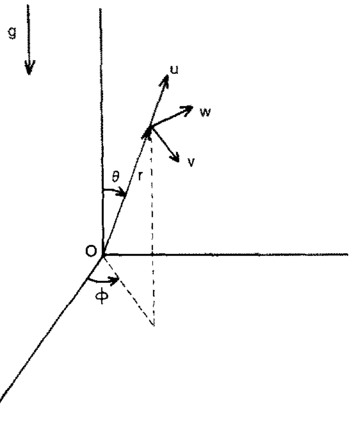

FIG. 1. The configuration of interest: a point source located at the center of a spherical-polar coordinate system in an unbounded porous medium.

$$\sigma = [\lambda(\rho c_p)_f + (1-\lambda(\rho c_p)_s]/(\rho c_p)_f \tag{6}$$

where subscripts f and s stand for fluid and solid matrix, respectively. It is assumed that the fluid and the solid matrix are in local thermal equilibrium. In buoyancy induced heat and mass transfer, the driving density gradient generally depends on both temperature and concentration. In accordance with the linear Boussinesq approximation the density is assumed constant everywhere except in the body force terms of the momentum equations (2, 3) where it is given by

$$\rho = \rho_\infty[1-\beta(T-T_\infty)+\beta_c(c-c_\infty)]. \tag{7}$$

In the above equation β and β_c are the coefficient of thermal expansion and the coefficient of concentration expansion respectively. Subscript ∞ indicates the properties at a reference state.

The governing equations were simplified by introducing the streamfunction ψ which satisfies the continuity equation identically

$$u = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{8, 9}$$

In addition, the pressure terms appearing in the momentum equations (2), (3) were eliminated by cross-differentiation to yield a unique momentum equation statement. Finally, the governing equations were placed in dimensionless form by defining a new set of variables

$$t_* = \frac{t\alpha}{K\sigma}, \quad r_* = \frac{r}{K^{1/2}}, \quad T_* = \frac{(T-T_\infty)kK^{1/2}}{q},$$
$$\psi_* = \frac{1}{\alpha K^{1/2}}, \quad c_* = \frac{(c-c_\infty)DK^{1/2}}{m}, \quad \psi_* = \frac{\psi}{\alpha K^{1/2}}. \tag{10}$$

The dimensionless governing equations are

$$\frac{1}{r_*^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi_*}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2\psi_*}{\partial r_*^2}$$
$$= Ra\left\{\left(\cos\theta\frac{\partial T_*}{\partial\theta}+r_*\sin\theta\frac{\partial T_*}{\partial r_*}\right)\right.$$
$$\left.-N\left(\cos\theta\frac{\partial c_*}{\partial\theta}+r_*\sin\theta\frac{\partial c_*}{\partial r_*}\right)\right\} \tag{11}$$

$$\frac{\partial T_*}{\partial t_*}+\frac{1}{r_*^2\sin\theta}\left(\frac{\partial\psi_*}{\partial\theta}\frac{\partial T_*}{\partial r_*}-\frac{\partial\psi_*}{\partial r_*}\frac{\partial T_*}{\partial\theta}\right)$$
$$=\frac{1}{r_*^2}\frac{\partial}{\partial r_*}\left(r_*^2\frac{\partial T_*}{\partial r_*}\right)+\frac{1}{r_*^2\sin\theta}$$
$$\times\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T_*}{\partial\theta}\right) \tag{12}$$

$$\gamma\frac{\partial c_*}{\partial t_*}+\frac{1}{r_*^2\sin\theta}\left(\frac{\partial\psi_*}{\partial\theta}\frac{\partial c_*}{\partial r_*}-\frac{\partial\psi_*}{\partial r_*}\frac{\partial c_*}{\partial\theta}\right)$$
$$=\frac{1}{Le}\left\{\frac{1}{r_*^2}\frac{\partial}{\partial r_*}\left(r_*^2\frac{\partial c_*}{\partial r_*}\right)+\frac{1}{r_*^2\sin\theta}\right.$$
$$\left.\times\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial c_*}{\partial\theta}\right)\right\} \tag{13}$$

where Ra is the Rayleigh number based on the heat strength of the source q and the permeability of the porous matrix

$$Ra = \frac{Kg\beta q}{\nu\alpha k} \tag{14}$$

γ and Le are the ratio of the matrix porosity divided by the heat capacity ratio, and the Lewis number respectively

$$\gamma = \frac{\lambda}{\sigma}, \quad Le = \frac{\alpha}{D}. \tag{15, 16}$$

Note that if $\gamma = Le = 1$, equations (12) and (13) take identical form. Hence, in this case, solutions for the temperature distribution are also solutions for the concentration distribution. Parameter N measures the relative importance of chemical and thermal diffusion in causing the density gradient which drives the flow

$$N = \frac{\beta_c(m/D)}{\beta(q/k)}. \tag{17}$$

It is worth discussing that N is zero for no species diffusion, infinite for no thermal diffusion, negative for both effects combining to drive the flow and positive for the two effects opposed.

Equations (11)–(13) are to be solved subject to the following initial conditions

$$u = v = 0, \quad T = T_\infty, \quad c = c_\infty \quad \text{at} \quad t = 0. \tag{18}$$

The boundary conditions of the problem are

$$u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad c \to c_\infty \quad \text{as} \quad r \to \infty$$
$$v = \frac{\partial u}{\partial\theta} = \frac{\partial T}{\partial\theta} = \frac{\partial c}{\partial\theta} = 0 \quad \text{at} \quad \theta = 0, \pi. \tag{19}$$

The origin of the system of coordinates is a singular point for velocity, temperature and concentration, since the heat and species source is located there. Therefore u, v, T and c are singular as $1/r$ in the limit $r \to 0$. For the temperature and concentration fields this behavior is described by a heat and a species balance over a spherical surface of radius approaching zero, containing the origin

$$q = \lim_{r\to 0}\left[-k(4\pi r^2)\frac{\partial T}{\partial r}\right],$$
$$m = \lim_{r\to 0}\left[-D(4\pi r^2)\frac{\partial c}{\partial r}\right]. \tag{20, 21}$$

3. SOLUTION FOR THE TRANSIENT STATE

In the limit of small Rayleigh numbers ($Ra \to 0$) the problem stated in the previous section accepts an approximate analytical solution in terms of power series expansions in Ra for ψ_*, T_* and c_*. Hence, we write

$$\begin{pmatrix}\psi_*\\ T_*\\ c_*\end{pmatrix} = \sum_{n=0}^{\infty}\begin{pmatrix}\psi_{*n}\\ T_{*n}\\ c_{*n}\end{pmatrix}Ra^n. \tag{22}$$

The successive terms ψ_{*_n}, T_{*_n}, c_{*_n} in equation (22) are obtained by combining (22) with the governing equations (11)–(13) and solving the equations resulting after collecting terms of equal power in Ra. For the sake of brevity we present only the final results.

The zeroth-order functions ψ_{*_0}, T_{*_0}, c_{*_0} correspond to the state of pure diffusion. Hence, we can take $\psi_{*_0} = 0$ and, according to Carslaw and Jaeger [14],

$$T_{*_0} = \frac{1}{4\pi r}\operatorname{erfc}\eta \tag{23a}$$

$$c_{*_0} = \frac{1}{4\pi r}\operatorname{erfc}(A\eta) \tag{23b}$$

where

$$\eta = \frac{r_*}{2t_*^{1/2}}, \quad A = (\gamma Le)^{1/2} \tag{24, 25}$$

and where

$\operatorname{erfc}\eta = 1-\operatorname{erfc}\eta$ with

$$\operatorname{erf}\eta = \frac{2}{\pi^{1/2}}\int_0^\eta \exp(-\xi^2)\,\mathrm{d}\xi. \tag{26, 27}$$

It can be seen that both the zeroth-order temperature and concentration distributions blow up as $1/r_*$ at the source ($r_* \to 0$).

The terms of order Ra in (11) yield the equation necessary to determine ψ_{*_1}. To separate variables we set

$$\psi_{*_1} = \frac{t_*^{1/2}}{2\pi}\sin^2\theta f(\eta) \tag{28}$$

where $f(\eta)$ satisfies the ordinary differential equation

$$\begin{aligned}\eta^2 f'' - 2f = &-\frac{2}{\pi^{1/2}}\eta^2\exp(-\eta^2) - \eta\operatorname{erfc}\eta \\ &+\frac{N}{A}\left\{\frac{2}{\pi^{1/2}}(A\eta)^2\exp[-(A\eta)^2]\right. \\ &\left.+(A\eta)\operatorname{erfc}(A\eta)\right\}.\end{aligned} \tag{29}$$

The general solution to equation (29) has the form

$$\begin{aligned}f = &\frac{c_1}{\eta} + c_2\eta^2 + \frac{\eta}{2}\operatorname{erfc}\eta + \frac{1}{4\eta}\operatorname{erf}\eta - \frac{1}{2\pi^{1/2}} \\ &\times\exp(-\eta^2) - \frac{N}{A}\left\{\frac{A\eta}{2}\operatorname{erfc}(A\eta)\right. \\ &\left.+\frac{1}{4A\eta}\operatorname{erf}(A\eta) - \frac{1}{2\pi^{1/2}}\exp[-(A\eta)^2]\right\}.\end{aligned} \tag{30}$$

The requirements that u and v approach zero as $r \to \infty$ and that u and v increase as $1/r$ in the limit $r \to 0$ (equation 1), combined with equations 8 and 9, yield $c_1 = c_2 = 0$. Hence, the final expression for ψ_{*_1} reads

$$\begin{aligned}\psi_{*_1} = &\frac{t_*^{1/2}}{2\pi}\sin^2\theta\left\{\frac{\eta}{2}\operatorname{erfc}\eta + \frac{1}{4\eta}\operatorname{erf}\eta - \frac{1}{2\pi^{1/2}}\right. \\ &\times\exp(-\eta^2) - \frac{N}{A}\left[\frac{A\eta}{2}\operatorname{erfc}(A\eta) + \frac{1}{4(A\eta)}\right. \\ &\left.\left.\times\operatorname{erf}(A\eta) - \frac{1}{2\pi^{1/2}}\exp[-(A\eta)^2]\right]\right\}.\end{aligned} \tag{31}$$

It is worth noting that for $N = 0$ result (31) reduces to the expression reported in ref. [10] where no species diffusion was considered. Figure 2 shows a set of transient streamlines $\psi_{*_1}/t_*^{1/2}(1-N) = \text{constant}$ for the special case $A = 1$. The flow field at small Rayleigh numbers consists of a circular vortex whose center is located at $\eta = 0.881$. The radius of the vortex increases in time as $t_*^{1/2}$ and it varies with parameter N as $(1-N)$. Negative values of N strengthen the flow and positive values of N weaken the flow. The bulging of the streamlines near the source is due to the impulsive effect of the source on the fluid particles near it, that is, the fluid particles coming from the lower half space 'converge' toward the source to replace the particles which have already moved upward, driven by the strong buoyancy force in the vicinity of the source. The existence of mass transfer induced buoyancy, affects the distance from the source within which this impulsive effect is felt. A more detailed illustration of this fact is shown in Fig. 5 for the $t_* \to \infty$ limit. This limit corresponds to the steady-state flow pattern.

In the case where the two buoyancy mechanisms are opposing ($N > 0$), parameter A has a striking effect on the transient flow field induced by the source. As illustrated in Fig. 3, values of A of order less than unity give life to a downward flow far from the source, surrounding the vortex near the source. This phenomenon makes sense physically if we realize that

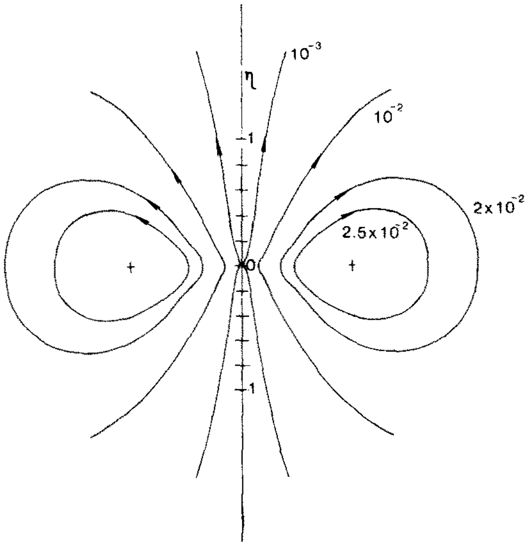

FIG. 2. Transient streamlines, $\psi_{*_1}/(1-N)t_*^{1/2} = \text{constant}$, around heat and chemical species source for $A = 1$.

parameter A represents the ratio of two length scales at early times, namely, the ratio of the thermal penetration length scale $r_T \sim 0(\sqrt{\alpha t/\sigma})$, (equation 4), to the species penetration length scale $r_c \sim 0(\sqrt{Dt/\lambda})$, (equation 5). Values of A less than unity imply that the chemical species diffuses faster than heat at early times. Hence, outside the region in which the thermal effect of the source is felt, $r_T \sim 0(\alpha t/\sigma)^{1/2}$, the chemical species concentration gradients act alone, creating a downward flow in the porous medium. Clearly, the phenomenon described above is a property of the transient state since in the steady state the heating effect of the sourse is felt throughout the porous space. The dashed line in Fig. 3 represents the streamline $(\psi_{*_1})/(t_*^{1/2}/2\pi) = 0$.

To obtain the first convective corrections for the temperature and concentration fields we need to combine equations (23a), (31) and equations (23b), (31) with equations (12) and (13), respectively. After separating variables by setting

$$T_{*_1} = \cos\theta G(\eta)/t_*^{1/2}, \quad c_{*_1} = \cos\theta F(\eta)/t_*^{1/2} \tag{32, 33}$$

we obtain the following ordinary differential equations for $G(\eta)$ and $F(\eta)$

$$\begin{aligned} &\eta^2 G'' + 2\eta(\eta^2+1)G' + 2(\eta^2-1)G \\ &= -\frac{1}{64\pi^2}\left[\frac{1}{\eta^2}\operatorname{erfc}\eta + \frac{2}{\eta\pi^{1/2}}\exp(-\eta^2)\right] \\ &\times\left\{2\eta\operatorname{erfc}\eta + \frac{1}{\eta}\operatorname{erf}\eta - \frac{2}{\pi^{1/2}}\exp(-\eta^2)\right. \\ &-\frac{N}{A}\left[2(A\eta)\operatorname{erfc}(A\eta) + \frac{1}{A\eta}\operatorname{erf}(A\eta)\right. \\ &\left.\left.-\frac{2}{\pi^{1/2}}\exp[-(A\eta)^2]\right]\right\} \end{aligned} \tag{34}$$

$$\begin{aligned} &(A\eta)^2F'' + 2(A\eta)[(A\eta)^2+1]F' \\ &+2[(A\eta)^2-1]F = -\frac{LeA^2}{64\pi^2}\left[\frac{1}{(A\eta)^2}\right. \\ &\left.\times\operatorname{erfc}(A\eta) + \frac{2}{(A\eta)\pi^{1/2}}\exp[-(A\eta)^2]\right] \\ &\times\left\{2\eta\operatorname{erfc}\eta + \frac{1}{\eta}\operatorname{erf}\eta - \frac{2}{\pi^{1/2}}\exp(-\eta^2)\right. \\ &-\frac{N}{A}\left[2(A\eta)\operatorname{erfc}(A\eta) + \frac{1}{(A\eta)}\right. \\ &\left.\left.\times\operatorname{erf}(A\eta) - \frac{2}{\pi^{1/2}}\exp[-(A\eta)^2]\right]\right\}. \end{aligned} \tag{35}$$

It is worth noting that in equation (35) the primes denote differentiation with respect to $(A\eta)$.

The general solutions for G and F are

$$G = \frac{1}{\eta^2}\exp(-\eta^2)\left\{B_1 + B_2\int_0^\eta x^2\exp(x^2)\,dx\right\} + G_p(\eta) \tag{36}$$

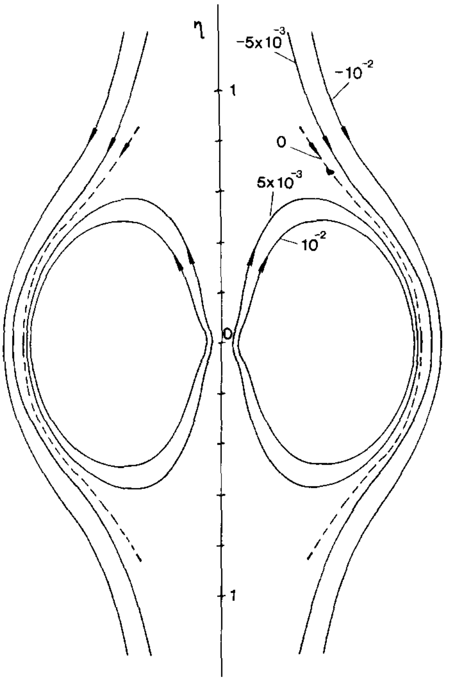

FIG. 3. The effect of parameter A on the transient streamlines, $(\psi_*)/(t_*^{1/2}/2\pi)$ = constant, for $A = 0.1$, $N = 0.5$.

$$F = \frac{1}{(A\eta)^2}\exp[-(A\eta)^2] \times\left\{B_3 + B_4\int_0^{A\eta} x^2\exp(x^2)\,dx\right\} + F_p(A\eta) \tag{37}$$

where $G_p(\eta)$ and $F_p(A\eta)$ are particular solutions which satisfy equations (34) and (35), respectively. Applying the boundary conditions at $\eta \to 0$ and $\eta \to \infty$, equations (19)–(21), yields $B_1 = B_2 = B_3 = B_4 = 0$. Hence, the first convective corrections for temperature and concentration are proportional to the particular solutions $G_p(\eta)$ and $F_p(A\eta)$, i.e.

$$(T_{*_1}, c_{*_1}) = \frac{\cos\theta}{t_*^{1/2}}(G_p(\eta), F_p(A\eta)). \tag{38}$$

We were able to obtain approximate expressions for $G_p(\eta)$ and $F_p(A\eta)$ valid in the important region $\eta < 1$, i.e. close to the point source. Proceeding as in ref. [10] we write

$$G_p = \sum_{n=-1}^{\infty} D_n\eta^n \tag{39}$$

$$F_p = \sum_{n=-1}^{\infty} E_n(A\eta)^n. \tag{40}$$

Equations (39), (40) take into account the fact that T_{*_1} and c_{*_1} are as singular as $1/\eta$ in the limit $\eta \to 0$. The coefficients D_n and E_n are obtained by also expanding the right hand sides of equations (34), (35) in power series in $\eta < 1$ and equating terms of equal power in η.

The expressions for the first nine coefficients for T_{*1} are

$$D_{-1} = \frac{1-N}{64\pi^2}, \quad D_0 = -\frac{1}{48\pi^{5/2}}(1-NA)$$

$$D_1 = 0, \quad D_2 = \frac{1}{96\pi^{5/2}}\left(\frac{NA^3}{5} - NA + \frac{9}{5} - N\right)$$

$$D_3 = -\frac{1}{180\pi^3}(1-NA), \quad D_4 = -\frac{1}{288\pi^{5/2}} \times \left(\frac{NA^5}{35} - \frac{2}{5}NA^4 + \frac{NA^3}{5} - NA - N + \frac{76}{35}\right)$$

$$D_5 = -\frac{1}{2520\pi^3}(NA^3 + 7NA - 8)$$

$$D_6 = -\frac{1}{320\pi^{5/2}}\left(-\frac{NA^7}{378} - \frac{NA^5}{126} + \frac{NA^4}{9} - \frac{NA^3}{18} + \frac{5}{18}NA + \frac{22}{63}N - \frac{127}{189}\right)$$

$$D_7 = -\frac{1}{22680\pi^3}(-NA^5 - 4.8NA^3 - 19NA + 24.8). \tag{41}$$

Similarly, the expressions for the first nine coefficients for c_{*1} are

$$E_{-1} = \frac{LeA}{64\pi^2}(1-N), \quad E_0 = -\frac{LeA}{48\pi^{5/2}}(A^{-1} - N)$$

$$E_1 = 0, \quad E_2 = \frac{LeA}{96\pi^{5/2}}\left(-\frac{A^{-3}}{5} + A^{-1} - \frac{9}{5}N + 1\right)$$

$$E_3 = -\frac{LeA}{180\pi^3}(A^{-1} - N),$$

$$E_4 = -\frac{LeA}{288\pi^{5/2}}\left[-\frac{A^{-5}}{35} - \frac{A^{-3}}{5} + A^{-1} - \frac{76}{35}N + \frac{7}{5}\right]$$

$$E_5 = -\frac{LeA}{2520\pi^3}[-A^{-3} - 7A^{-1} + 8N]$$

$$E_6 = -\frac{LeA}{320\pi^{5/2}}\left[\frac{A^{-7}}{378} + \frac{A^{-5}}{126} + \frac{A^{-3}}{18} - \frac{5}{18}A^{-1} + \frac{127}{189}N - \frac{29}{63}\right]$$

$$E_7 = -\frac{LeA}{22680\pi^3}(A^{-5} + 4.8A^{-3} + 19A^{-1} - 24.8N). \tag{42}$$

It should be pointed out that the above procedure is valid if $\eta < 1$ and also $A\eta < 1$. This implies that parameter A must be $0(1)$ or less. This range of values for A is appropriate for a variety of physical and laboratory applications. To exemplify, we state two such applications:

—Diffusion of methanol and hydrogen in water-$\gamma \approx 0.2$, $Le \approx 100$, hence $A \approx 2$.

—Diffusion of methanol and hydrogen in water-saturated glass beads: $\gamma \approx 1$, $Le \approx 0.3$, hence $A \approx 0.3$.

The effect of parameter N on the radial distribution of T_{*1} and c_{*1} is illustrated in Fig. 4 for the case $Le = A = 1$. When the two mechanisms combine to drive the flow ($N = -0.5$), the impact of the source on the temperature and concentration distributions in the porous medium is felt at larger distances away from the source. In the opposite case ($N = 0.5$), the convective contribution to the temperature field weakens relative to the 'no species generation' limit ($N = 0$). This phenomenon was expected however, for the generated substance now acts as a brake on the thermally induced flow.

It is worth noting that in the region of net upward flow, equation (38) dictates that the first convective corrections T_{*1} and c_{*1} to the temperature and concentration fields produce an increase in temperature and species concentration for points in the upper half space ($0 \leqslant \theta \leqslant \pi/2$) accompanied by an equal decrease in the temperature and concentration in the lower half space. On the other hand, if an isothermal region of downward flow, like the region shown in Fig. 3, exists, then the lower half space in this region is richer in chemical species than the upper half space.

The complexity of the expressions obtained for ψ_{*1}, T_{*1}, and c_{*1} makes it clear that the perturbation analysis for the transient problem is difficult to handle beyond the first order convective effect because the algebra becomes very tedious. However, in the steady state ($t_* \to \infty$) the analysis is somewhat simpler, since time does not enter the problem as a variable. The steady-state regime is examined in the next section.

4. THE STEADY STATE

As implied by equations (23a), (23b), in the limit $t_* \to \infty$ the temperature and concentration fields in the purely diffusive regions are spherically symmetric and they decrease as $1/r_*$ away from the source. In the presence of natural convection, these fields will be distorted to a degree dictated by the strength of the buoyancy driven flow. If the net flow is directed upwards it is reasonable to expect the points in the porous medium located in the upper half-space to be warmer and of higher species concentration than the points in the lower half space. The strength of the flow, which is responsible for this fact, is determined by the nature and the relative magnitude of the heat and mass transfer in the buoyancy term of the momentum equation (11).

To determine ψ_*, T_*, and c_* in the steady state, we follow a procedure identical to the one described previously for the transient state. For brevity, we show here only the final expressions for streamfunction temperature and concentration up to the second order convective correction

$$\psi_* = \frac{r_*}{8\pi}\left[(1-N)\sin^2\theta\, Ra + \frac{(1-N)(1-NLe)}{24\pi} \times \sin\theta \sin 2\theta\, Ra^2 + \cdots\right] \tag{43}$$

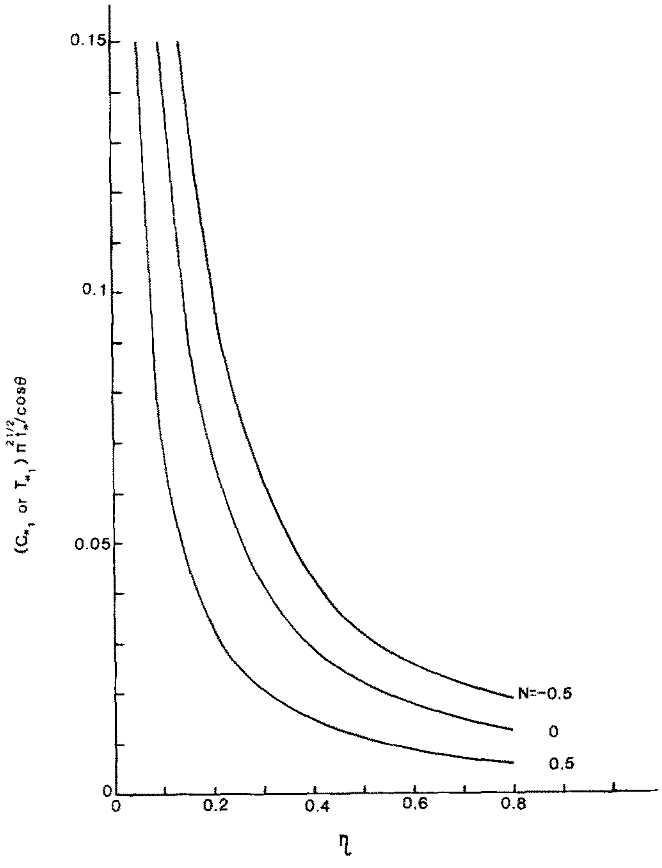

FIG. 4. The effect of parameter N on the radial profile of the first-order convective correction for temperature $(T_{*_1}\pi^2 t_*^{1/2}/\cos\theta)$ or concentration $(c_{*_1}\pi^2 t_*^{1/2}/\cos\theta)$ for $A = Le = 1$.

$$T_* = \frac{1}{4\pi r_*}\left[1 + \frac{1-N}{8\pi}\cos\theta\, Ra + \left(\frac{(1-N)^2}{256\pi^2} + \frac{(1-N)(1-NLe)}{384\pi^2}\right)\cos 2\theta\, Ra^2 + \cdots\right] \quad (44)$$

$$c_* = \frac{1}{4\pi r_*}\left[1 + Le\frac{1-N}{8\pi}\cos\theta\, Ra + Le\left(\frac{(1-N)^2}{256\pi^2} + \frac{(1-N)(1-NLe)}{384\pi^2}\right)\cos 2\theta\, Ra^2 + \cdots\right]. \quad (45)$$

The steady-state flow field is illustrated in Fig. 5 for $Le = 1$ and $Ra = 5$. The curves in this figure correspond to lines of $\psi_* = Ra\, R_*/8\pi$, where R_* is a fixed radial distance from the source. It is clear that the creation of chemical species can strengthen ($N = -0.5$) or weaken ($N = 0.5$) the steady-state flow created by thermal buoyancy ($N = 0$). The effect of the natural convection on the temperature and concentration fields is shown in Fig. 6. The curves in this figure correspond to lines of $T_* = 1/4\pi R_*$ or lines of $c_* = 1/4\pi R_*$, since for this illustration also, we fixed the Lewis number $Le = 1$. Once again, as the nature of species generation changes from slowing down the flow to aiding the flow, the warm, high concentration region shifts upwards. Finally, increasing the Lewis number while keeping the

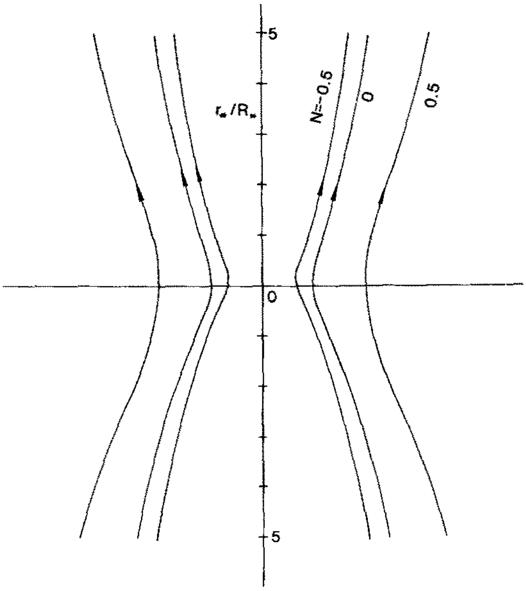

FIG. 5. Steady-state streamline pattern for $Le = 1, Ra = 5$. The curves represent the streamline $\psi_* = RaR_*/8\pi$ for increasing values of N.

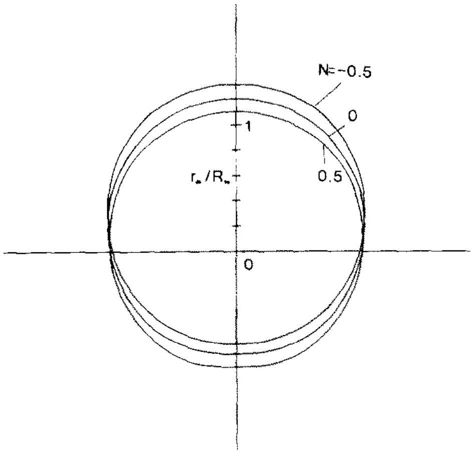

FIG. 6. Steady-state temperature and concentration fields for $Le = 1, Ra = 5$. The lines represent the isotherm $T_* = 1/4\pi R_*$ (or the curve $c_* = 1/4\pi R_*$) for increasing values of N.

rest of the parameters constant ($N = 0.5$, $Ra = 5$) as shown in Fig. 7, shifts the high concentration region upwards while inducing a weak change on the temperature field. Indeed, the change on the isotherm $T_* = 1/4\pi R_*$ as the Lewis number increases from 0.5 to 1.5, is invisible.

5. CONCLUSIONS

In this paper we presented a theoretical study of natural convection from a heat and chemical species generating point source imbedded in an infinite porous medium. The study aimed to extend the results reported in ref. [10] which was concerned with the phenomenon of natural convection from a point source in a porous medium, induced by thermal buoyancy alone, i.e. in the absence of species generation. Both the transient and the steady-state solutions for the temperature, the concentration and the flow fields were obtained by means of perturbation analysis in the thermal Rayleigh number. It is worth noting that the broader problem of double diffusion is normally dependent on two Rayleigh numbers: one based on the thermal field and the other based on the concentration field. The present study, for simplicity, was formulated so that the contribution of the species concentration gradients in driving the flow was measured by the magnitude of parameter N (equation 17). Hence, the findings of this report determine the impact of species concentration gradients upon the thermally driven flow.

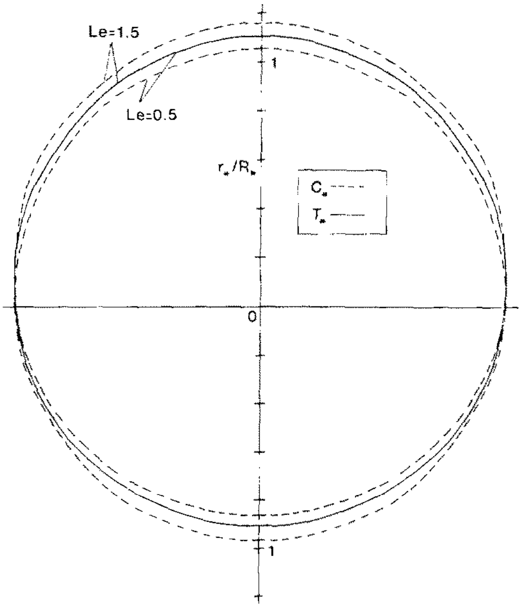

FIG. 7. The effect of the Lewis number, Le, on the steady-state isotherms ($T_* = 1/4\pi R_*$) and the lines of constant concentration ($c_* = 1/4\pi R_*$) for $Ra = 5$, $N = 0.5$.

The results of this study are valid in the diffusion-dominated regime. In other words, the range of values of the main parameters of the problem (Ra, N, A, Le) for which the findings of this study are reasonably accurate, can be determined by requiring that the convective contributions to the expressions for the temperature and concentration fields are considerably smaller than the pure diffusive contribution ($Ra = 0$). This criterion was satisfied for all the illustrations in the course of the study (Figs. 2–7), that is, the values of Ra, N, A and Le used for these illustrations yielded small convective 'corrections' to the pure diffusion-dominated temperature and concentration fields. Further verification of the range of validity of the perturbation analysis should be sought via numerical or experimental solutions. In the special case of absent chemical species generation ($N = 0$) where the Rayleigh number is the only important parameter of the problem and where the expressions are simpler, the solution yields approximately accurate results for up to $Ra = 10$.

In summary, the generation of a substance simultaneously with heat generation from a concentrated source can seriously affect the properties of the buoyancy induced flow which penetrates the infinite porous medium. A number of new parameters (N, Le, A) entered the problem due to the existence of species concentration gradients. The effect of these parameters in altering the transient characteristics of the problem was illustrated in Figs. 2–4. Special attention should be given to the fact that a down-flow surrounding the vortex near the source may exist in the case where the two buoyancy mechanisms are opposing ($N > 0$) as shown in Fig. 3. The steady-state flow, temperature, and concentration fields (Figs. 5–7) are clearly affected by the aiding or retarding effect of the presence of a substance in the fluid saturating the porous medium. If the nature of this substance is to aid the thermal flow ($N < 0$) the bulging of the streamlines near the source becomes more obvious (Fig. 5). Varying the Lewis number has a stronger impact on the concentration field than it has on the temperature field (Fig. 7).

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TRANSFERT DE CHALEUR ET DE MASSE INDUIT PAR GRAVITE A PARTIR D'UNE SOURCE CONCENTREE DANS UN MILIEU POREUX INFINI

Résumé—On considère l'étude analytique de la convection naturelle de chaleur et de masse, induite par une source concentrée qui est située dans un milieu poreux infini. Les champs stationnaires variables et stationnaires de température et de concentration sont obtenus par des développements en série en termes de nombre de Rayleigh basé sur la perméabilité du milieu poreux et le flux de chaleur issu de la source. Le rôle des espèces chimiques créées par la source est d'aider ou de retarder l'écoulement induit par la gravité thermique. On fournit dans le cours de l'étude des relations déterminant l'effet de la génération d'espèce sur les champs variables ou stationnaires de température dans le milieu poreux. Les discussion sont focalisées sur le cas où l'écoulement net est ascendant. Même si la chaleur est spécifiée être un des deux mécanismes de diffusion, les résultats de cette étude s'appliquent au cas de l'écoulement induit par la gravité à partir d'une source concentrée générant simultanément deux composants chimiques différents.

WÄRME- UND STOFFTRANSPORT IN AUFTRIEBSSTRÖMUNGEN AN EINER PUNKTFÖRMIGEN QUELLE IN EINEM UNENDLICHEN PORÖSEN MEDIUM

Zusammenfassung—Diese Arbeit berichtet über eine analytische Untersuchung des Wärme- und Stoffübergangs bei natürlicher Konvektion, der durch eine punktförmige, in einem unendlichen Medium befindliche Quelle hervorgerugen wird. Für stationäre und instationäre Strömung werden Temperatur und Konzentrationsfelder in Reihenentwicklungen der Rayleigh-Zahl in Abhängigkeit von der Permeabilität des porösen Mediums und der Wärmeerzeugungsrate der Quelle angegeben. Eine von der Quelle abgegebene chemische Substanz kann die durch thermische Auftriebskräfte hervorgerufene Strömung entweder beschleunigen oder verzögern. Im Laufe der Studie werden Ausdrücke vorgestellt, die den Einfluß der Substanzfreisetzung auf die instationären und stationären Temperatur- und Strömungsfelder im porösen Medium beschreiben. Alle Erörterungen in dieser Arbeit beziehen sich auf den Fall einer Netto-Aufwärtsströmung. Obwohl der Wärmetransport als einer der beiden Diffusionsmechanismen angenommen wurde, lassen sich die Ergebnisse dieser Studie auch auf den Fall einer Auftriebsströmung von einer punktförmigen Quelle, die gleichzeitig zwei unterschiedliche chemische Komponenten erzeugt, anwenden.

О ТЕПЛО-И МАССОПЕРЕНОСЕ, ВЫЗВАННОМ ПОДЪЕМНОЙ СИЛОЙ ОТ ТОЧЕЧНОГО ИСТОЧНИКА В БЕСКОНЕЧНОЙ ПОРИСТОЙ СРЕДЕ

Аннотация—В работе дано аналитическое исследование естественно-конвективного тепло-и массопереноса, инициированного точечным источником, помещенным в бесконечную пористую среду. Переходное и стационарное течения, поля температуры и концентрации представлены в виде рядов разложений по числу Рэлея, построенных на основе проницаемости пористой среды и скорости тепловыделения из источника. Воздействие химического состава, создаваемое источником, либо способствует, либо препятствует течению, вызванному тепловой подъемной силой. Даны полученные в результате исследования выражения, определяющие влияние генерации химического состава на результирующие температурные и скоростные поля для переходного и установившегося течений в пористой среде. Исследование сосредоточено на случае, когда основное течение направлено вверх. Если даже теплоперенос осуществляется одним из двух механизмов диффузии, то результаты работы справедливы также для случая течения, инициированного подъемной силой от точечного источника, выделяющего одновременно две различных химических компоненты.